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THE SYMBOLIC DYNAMICS OF TILING THE INTEGERS*

BY

ETHAN M. COVEN

Department of Mathematics, Wesleyan University
Middletown, CT 06459-0128, USA
e-mail: ecoven@wesleyan.edu

AND

WILLIAM GELLER

Department of Mathematics, Indiana University-Purdue University at Indianapolis
Indianapolis, IN 46202-3216, USA
e-mail: wgeller@math.iupui.edu

AND

SYLVIA SILBERGER

Department of Mathematics, Hofstra University
Hempstead, NY 11549, USA
e-mail: matsbs@hofstra.edu

AND

WILLIAM THURSTON

Department of Mathematics, University of California Davis
Davis, CA 95616, USA
e-mail: wpt@math.ucdavis.edu

ABSTRACT

A finite collection $\mathcal{P}$ of finite sets tiles the integers iff the integers can be expressed as a disjoint union of translates of members of $\mathcal{P}$. We associate with such a tiling a doubly infinite sequence with entries from $\mathcal{P}$. The set of all such sequences is a sofic system, called a tiling system. We show that, up to powers of the shift, every shift of finite type can be realized as a tiling system.

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1. Introduction

For notation, terminology, and basic results of symbolic dynamics, see the book by D. Lind and B. Marcus [LM].

Let $\mathcal{P} = \{P_1, \ldots, P_K\}$ be a finite collection of finite subsets of the integers $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$, called **prototiles**. We normalize the prototiles so that each has minimum 0. A **tile** is a translate of a prototile. If the integers can be expressed as a disjoint union of tiles, $\mathbb{Z} = \bigcup(t_j + P_{k_j})$, we say that $\mathcal{P}$ **tiles the integers.** The corresponding **tiling of the integers** by $\mathcal{P}$ is the point $x = (x_i) \in \prod_{i=-\infty}^{\infty}\{1, 2, \ldots, K\}$ defined by $x_i = k$ if and only if there exists j such that $i \in t_j + P_{k_j}$ and $k_j = k$. Thus we can think of a tiling as being given by a bi-infinite sequence of colors, where the colors are in one-to-one correspondence with the prototiles.

Let σ denote the **shift**, $(\sigma(x))_i = x_{i+1}$. The collection of points corresponding to tilings of the integers by $\mathcal{P}$, denoted $T(\mathcal{P})$, is closed and shift-invariant. We call $\sigma\colon T(\mathcal{P}) \to T(\mathcal{P})$ a **tiling system**. We first show that every tiling system is sofic. We then prove our main result: up to powers of the shift, every shift of finite type can be realized as a tiling system.

MAIN THEOREM: *Let $\sigma\colon \Sigma \to \Sigma$ be a shift of finite type. Then there is a positive integer m and a tiling system $\sigma\colon T \to T$ such that*

(1) *$T = T_0 \cup T_1 \cup \cdots \cup T_{m-1}$, where the T_i are closed and cyclically permuted by the shift σ;*

(2) *$\sigma^m\colon \Sigma \to \Sigma$ is topologically conjugate to every $\sigma^m\colon T_i \to T_i$.*

Recall that a **Perron number** is a positive real algebraic integer that dominates its algebraic conjugates.

COROLLARY: *The set of topological entropies of tiling systems is the same as that of shifts of finite type, i.e., the set of logarithms of roots of Perron numbers.*

Remark: In the sequel we will sometimes, as is common in symbolic dynamics, call the space T a tiling system, the space Σ a shift of finite type, etc.

We note that partial results on nonemptiness of a tiling system generated by a single prototile are known. D. Newman [N] gave a simple number-theoretic criterion for determining whether or not a single prototile of prime power cardinality can be used to tile the integers. The first author and A. Meyerowitz [CM] found a related criterion which they showed to be sufficient for an arbitrary single prototile to tile the integers, and necessary if the cardinality is the product of two prime powers; its necessity for an arbitrary single prototile is unknown.

We are not aware of any results in this direction for tiling systems arising from multiple prototiles.

2. Tiling systems are sofic

Consider the following examples.

(1) $\mathcal{P} = \{\{0\}, \{0,1\}\}$. It is more convenient to think of $\mathcal{P}$ as $\{R, BB\}$, (R=red, B = blue). Then $T(\mathcal{P})$ is the set of all bi-infinite indexed concatenations of R and B such that between any two consecutive occurrences of R there is an even number of B's, the well-known even system. In this case $T(\mathcal{P})$ is also a renewal system, although we do not use that fact here. Recall that a **renewal system** is the collection of indexed bi-infinite sequences which are concatenations of a finite set of finite words from some alphabet. In the sequel, we shall abuse notation and write $T(R, BB)$ in place of $T(\{\{0\}, \{0,1\}\})$.

(2) $\mathcal{P} = \{\{0\}, \{0,2\}\}$, which we replace by $\{R, B \_ B\}$. Then $T(\mathcal{P})$ is the renewal system generated by words R, BRB, and $BBBB$.

The next example shows that not every subshift that is topologically conjugate to a tiling system is itself a tiling system.

(3) $\mathcal{P} = \{\{0\}, \{0,2,3\}\}$, or $\{R, B \_ BB\}$. $T(\mathcal{P})$ is the renewal system generated by R and $BRBB$. It is topologically conjugate to the renewal system generated by R and $BRBY$, but the latter is not a tiling system.

(4) $\mathcal{P} = \{R, BB \_ B, Y \_ \_ Y\}$, i.e., $\{\{0\}, \{0,1,3\}, \{0,3\}\}$. In this case $T(\mathcal{P})$ is not a renewal system. (Otherwise, consider the renewal word W which includes the first B in $\ldots RRRBBYBYYY \ldots \in T(\mathcal{P})$. But no such W can exist, since some R^n must be a renewal word and $\ldots RRRWRRR \ldots \notin T(\mathcal{P})$.)

To show that every tiling system is sofic, recall the proof that the even system $\sigma\colon T(R, BB) \to T(R, BB)$ is sofic — it is the image of the shift of finite type $\sigma\colon \widetilde{T}(R_1, B_1B_2) \to \widetilde{T}(R_1, B_1B_2)$ under the "drop the subscripts" map. Here $\widetilde{T}(R_1, B_1B_2)$ is the set of all bi-infinite indexed concatenations of R_1 and B_1B_2. We show that every "subscripted tiling system" is a shift of finite type. Clearly every tiling system can be obtained from a subscripted tiling system by dropping the subscripts.

Formally, let $\mathcal{P} = \{P_1, \ldots, P_K\}$ be a finite collection of prototiles. Write

$$P_k = \{0 = p_{k,1} < p_{k,2} < \cdots < p_{k,\ell_k}\}$$

and define $\widetilde{T} = \widetilde{T}(\mathcal{P})$, on alphabet $\{(k,\ell) : 1 \le k \le K, 1 \le \ell \le \ell_k\}$, by $x \in \widetilde{T}$ iff there is a tiling of the integers by members of $\mathcal{P}$, $\mathbb{Z} = \bigcup(t_j + P_{k_j})$, such that for every i, there exist $j = j(i)$ and $\ell = \ell(i)$ such that $i \in t_j + P_{k_j}$ and $x_i = (k_j, \ell)$.

Equivalently, $x \in \prod\{(k,\ell) : 1 \le k \le K, 1 \le \ell \le \ell_k\}$ is in $\widetilde{T}$ if and only if for every i, $x_i = (k,\ell)$ and $1 \le \ell' \le \ell_k$ imply $x_{i+p_{k,\ell'}-p_{k,\ell}} = (k,\ell')$. Informally, if x_i is an element of a tile, then the other elements of that tile appear in the appropriate places of x.

The following result was proved in conversations with K. Schmidt in Warwick in 1994.

THEOREM: *Every "subscripted tiling system" is a shift of finite type.*

Proof: Let L be the length of a longest prototile in $\mathcal{P}$. (For example, $B\,\_\,B$ has length 3.) We show that $\widetilde{T} = \widetilde{T}(\mathcal{P})$ is a shift of finite type by showing that if $x \in \prod\{(k,\ell)\}$ and every solid L-word which appears in x appears in some point of $\widetilde{T}$, then $x \in \widetilde{T}$.

Suppose that every solid L-word which appears in x appears in some $y \in \widetilde{T}$. Let $x_i = (k,\ell)$. Since $p_{k,\ell_k} + 1 \le L$, there exists $y \in \widetilde{T}$ such that

$$y_{i-p_{k,\ell}}, \ldots, y_{i-p_{k,\ell}+p_{k,\ell_k}} = x_{i-p_{k,\ell}}, \ldots, x_{i-p_{k,\ell}+p_{k,\ell_k}}.$$

But $y \in \widetilde{T}$ and $y_i = (k,\ell)$, so $y_{i-p_{k,\ell}+p_{k,\ell'}} = (k,\ell')$ for $1 \le \ell' \le \ell_k$. Hence $x \in \widetilde{T}$.

Informally, suppose that every solid L-word which appears in x appears in a subscripted tiling. Since no tile is longer than L, if x_i is an element of a tile, then the other elements of that tile appear in the appropriate places of x. Therefore $x \in \widetilde{T}$. ∎

COROLLARY: *Every tiling system is sofic.*

Remark: We cannot use $L - 1$ in the proof of the theorem. Again let $\mathcal{P} = \{R, BB\}$, so $\widetilde{T} = \widetilde{T}(R_1, B_1B_2)$ and $L = 2$. Every 1-word appearing in $x = \ldots B_1B_1B_1\ldots$ appears in some point of $\widetilde{T}$, but $x \notin \widetilde{T}$.

Not every sofic system can be realized as a tiling system, as is shown by the following

PROPOSITION: *A tiling system which has a point of period 2 must have at least two fixed points.*

By period here we will mean least period.

Proof: The point of period 2 is $\ldots abab\ldots$, so there are two prototiles, each consisting entirely of even integers. Each tiles the integers, so both $\ldots aaa\ldots$ and $\ldots bbb\ldots$ are in the tiling system. ∎

Similarly, if a tiling system has a point of period 3 or one of period 4, then it must have at least one fixed point. The existence of a point of period greater than 4 does not imply the existence of a fixed point.

We can also use the preceding proposition to show that powers of tiling systems need not even be conjugate to tiling systems:

(5) Consider $T = T(\mathcal{P})$ where $\mathcal{P} = \{RR\_R, B\_\_\_B\}$, i.e., $\{\{0,1,3\},\{0,4\}\}$. The tiling system $\sigma: T \to T$ has exactly one orbit of period 4 – $(RRBR)^\infty$, no points of period 2, and exactly one fixed point – B^∞. Its square, $\sigma^2: T \to T$, has a point of period 2 – $[(RR)(BR)]^\infty$, but only one fixed point – $(BB)^\infty$. So it is not topologically conjugate to a tiling system.

3. The main theorem

MAIN THEOREM: *Let $\sigma: \Sigma \to \Sigma$ be a shift of finite type. Then there is a positive integer m and a tiling system $\sigma: T \to T$ such that*

(1) *$T = T_0 \cup T_1 \cup \cdots \cup T_{m-1}$, where the T_i are closed and cyclically permuted by the shift σ;*

(2) *$\sigma^m: \Sigma \to \Sigma$ is topologically conjugate to every $\sigma^m: T_i \to T_i$.*

Proof: We may assume that $\Sigma = \Sigma_A$, the edge shift determined by a matrix A with nonnegative integer entries. A is the adjacency matrix of a directed graph G. The alphabet of Σ_A is the set of arcs (directed edges) of G, and $(x_i) \in \Sigma_A$ if and only if for every i, the terminal vertex of x_i is the initial vertex of x_{i+1}.

For every positive integer m, $\sigma^m: \Sigma_A \to \Sigma_A$ is topologically conjugate to $\sigma: \Sigma_{A^m} \to \Sigma_{A^m}$. We find a tiling system $\sigma: T \to T$ and a positive integer m such that $T = T_0 \cup T_1 \cup \cdots \cup T_{m-1}$, the T_i are closed and cyclically permuted by the shift, and $\sigma: \Sigma_{A^m} \to \Sigma_{A^m}$ is topologically conjugate to $\sigma^m: T_0 \to T_0$. Hence $\sigma^m: \Sigma_A \to \Sigma_A$ is topologically conjugate to every $\sigma^m: T_i \to T_i$.

Suppose that A is $V \times V$. Choose $n > V$ so that

$$(V \max A_{ij})^{13n} < (n+1)!.$$

Let $m = 13n$. Then every entry of $A^m = A^{13n}$ can be written (uniquely) as

$$c_1(1!) + c_2(2!) + \cdots + c_n(n!),$$

where $0 \le c_k \le k$ for $1 \le k \le n$.

We now construct the tiling system. The prototiles will be of two types: **barbells** and **racks** (to hold barbells). We will use the same terms for the corresponding tiles. In the sequel we will use colors to label prototiles. The symbols a, a' will stand for generic colors.

The barbells are the broken words of the form

$$a^2 \overset{\leftarrow 2r+1 \rightarrow}{\_\cdots\_} a^2$$

for $0 \le r \le 2n-2$.

The racks are chosen from the broken words of the form HCT of length $13n + 2J$, $1 \le J \le V$, where the **head** is

$$H = (a\ \_)^I a^{2n-2I}$$

for some I, $1 \le I \le V$; the **tail** is

$$T = (\_\ a)^J$$

for some J, $1 \le J \le V$; and the **center** is

$$C = a^{3n+i} \overset{\leftarrow 2k \rightarrow}{\_\cdots\_} a \overset{\leftarrow 2k \rightarrow}{\_\cdots\_} a^{8n-4k-1-i}$$

for some i and k, $0 \le i \le k-1$ and $1 \le k \le n$.

Given I, J with $1 \le I, J \le V$, write

$$(A^{13n})_{IJ} = c_1(1!) + c_2(2!) + \cdots + c_n(n!),$$

where $0 \le c_k \le k$ for $1 \le k \le n$. If $c_k \ne 0$, choose the racks to be the $c_k = c_k(I,J)$ broken words of the form

$$\left[(a\ \_)^I a^{2n-2I}\right] \left[a^{3n+i} \overset{\leftarrow 2k \rightarrow}{\_\cdots\_} a \overset{\leftarrow 2k \rightarrow}{\_\cdots\_} a^{8n-4k-1-i}\right] \left[(\_\ a)^J\right]$$

for $0 \le i \le c_k - 1$.

The barbells and racks have the following properties.

- The head $H = (a\ \_)^I a^{2n-2I}$ of a rack can be filled by the tail $T = (\_\ a')^J$ of a rack in a tiling if and only if $I = J$.
- Label the blanks in the center

$$C = a^{3n+i} \overset{\leftarrow 2k \rightarrow}{\_\cdots\_} a \overset{\leftarrow 2k \rightarrow}{\_\cdots\_} a^{8n-4k-1-i}$$

of a rack by $\{1, 2, \dots, 4k\}$. Barbells can appear in a tiling only in the gaps in the centers of racks, starting only in odd places and straddling the a. Furthermore, the blanks in this center can be tiled by barbells in exactly $k!$ ways. To see this, define a permutation π of $\{1, 2, \dots, k\}$ by $\pi(j) = \ell$ if and only if a barbell $a'a' \overset{\leftarrow 2r+1 \rightarrow}{\_\cdots\_} a'a'$ occupies places labelled $2j-1, 2j, 2(k+\ell)-1, 2(k+\ell)$.
- The heads of racks can appear in a tiling starting only at places which differ by multiples of $13n$.

Let T be the tiling system with prototiles the barbells and racks chosen above. Then $T = T_0 \cup T_1 \cup \ldots \cup T_{13n-1}$, where T_i is the set of indexed bi-infinite sequences in T in which the heads appear starting at places congruent to i modulo $13n$. Thus T_0 consists of all indexed bi-infinite concatenations of words of length $13n$, of the form $\bar{H}\bar{C}$, starting at multiples of $13n$, where H and C are the head and center of a rack, and $\bar{H}$ and $\bar{C}$ are the solid words resulting from filling them in a tiling. Recall that if $H = (a\ \_)^I a^{2n-2I}$, then it must be filled by a tail $T = (\_\ a')^I$. C can be filled only by barbells.

Define an edge shift as follows. Let G' be the directed graph with vertices $1, 2, \ldots, V$, and an arc from I to J for each rack with head $(a\ \_)^I a^{2n-2I}$, tail $(\_\ a)^J$, and center tiled by barbells. There are $(A^{13n})_{IJ}$ arcs from I to J, and an arc with head $(a\ \_)^I a^{2n-2I}$ can follow an arc with tail $(\_\ a')^J$ if and only if $I = J$. Therefore since $m = 13n$, the adjacency matrix of G' is A^m and $\sigma^m\colon T_0 \to T_0$ is topologically conjugate to $\sigma\colon \Sigma_{A^m} \to \Sigma_{A^m}$, which in turn is topologically conjugate to $\sigma^m\colon \Sigma_A \to \Sigma_A$. ∎

It follows immediately from the Main Theorem that every topological entropy of a shift of finite type occurs as the entropy of a tiling system. Since the topological entropies of shifts of finite type and also of sofic systems are both equal to the logarithms of positive spectral radii of nonnegative integral matrices, or equivalently, the logs of roots of Perron numbers, we have

COROLLARY: *The set of topological entropies of tiling systems is the same as that of shifts of finite type, i.e., the set of logarithms of roots of Perron numbers.*

The main question we have considered has a natural analogue in $d > 1$ dimensions: can every $\mathbb{Z}^d$ shift of finite type be realized, up to a power, as a $\mathbb{Z}^d$ tiling system? Forthcoming work of the first author, A. Johnson, N. Jonoska, and K. Madden adapts the methods used here to address the higher-dimensional case.

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